Manipulating Bose-Einstein condensed atoms in toroidal traps

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We consider Bose-Einstein condensed atoms confined in a toroidal trap. We demonstrate that under conditions of one-dimensional behavior, the density distribution of the atoms may be exponentially localized/delocalized, even for very small variations in the trapping potential along the torus. This observation allows one to control the atom density externally via slight modifications of the trapping potential. For similar reasons, small irregularities of the trap may also have a very pronounced effect on the density of the cloud.

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I. INTRODUCTION

One of the many interesting consequences of recent experimental advances in the field of cold atoms is the fact that it is now possible to realize states of reduced dimensionality. More precisely, so long as the energy of the atoms due to their interactions is much smaller than the spacing of the energy levels associated with motion along some symmetry axis of the trap, the atomic state is dominated by the lowest state of the trapping potential, and this degree of freedom is effectively frozen. Thus, oblate/elongated traps can be used to achieve quasi-two/quasi-one dimensional behavior. As reported in Ref. [1], Ketterle and co-workers have achieved quasi-one-dimensional conditions. Using a variety of sophisticated techniques, experimentalists have produced traps spanning a wide range of other geometries, as reported in, e.g., Refs. [2, 3, 4, 5, 6, 7]. Several experimental and theoretical groups have also studied quasi-one-dimensional Bose-Einstein condensates in magnetic waveguides [8]. In a detailed study, Leboeuf and Pavloff have considered the effects of perturbing obstacles on the propagation of Bose-Einstein condensed atoms through such magnetic waveguides [9].

In the present study we focus on tight toroidal trapping potentials, which have been realized in various laboratories [4, 5, 6, 7]. Such traps can be used to create one-dimensional systems satisfying periodic boundary conditions. Since such systems have been under investigation for decades, it is clear that they are interesting from a theoretical point of view. Further, they may have numerous technological applications.

In a one-dimensional toroidal trap, the atomic density is homogeneous if the effective interaction between the atoms is repulsive [10]. It has been shown, however, that for a sufficiently strong attractive interaction, the atoms undergo a second-order quantum phase transition and form a localized density distribution [11, 12]. (This stands in contrast to a three-dimensional gas, which simply collapses.) In the inhomogeneous phase the atoms benefit from the formation of a localized "blob", which lowers their interaction energy.

The issue to be considered here is the effect of a controlled change in the trapping potential $V(\theta)$ along the torus as a consequence of small transverse irregularities. We will see that even small irregularities can materially alter this simple picture of localization. The effects to be considered were first encountered in early studies of radar. Hot spots were observed wherever waveguides were bent. It was soon recognized that any such nonuniformity in a quasi one-dimensional waveguide necessarily leads to exponentially localized, subthreshold resonances (i.e., bound states) with a correspondingly high local energy density.

In Sec. II we describe the problem to be considered in greater detail, and we present our model in Sec. III. Section IV contains our results for the ground state and the excitation spectrum of the gas. Section V contains a discussion of our results and presents the conclusions of this study.

II. EFFECT OF A DISTORTED TOROIDAL POTENTIAL

One might imagine that small irregularities in the one-dimensional potential, $V(\theta)$, do not have any substantial effect. The situation is very different in the quasi one-dimensional systems considered here. Consider first a free particle moving in an infinitely long waveguide of small transverse size. The wave function for this particle can be approximated as the product of the lowest transverse eigenfunction and a function of the distance along the waveguide. The associated eigenvalue of the transverse Hamiltonian provides an effective potential for the resulting one-dimensional motion. A localized broadening (or, equivalently, a bend) of an otherwise uniform waveguide will thus result in an attractive effective potential. It is familiar from elementary quantum mechanics that every purely attractive potential in one dimension has at least one exponentially bound state. The ubiquity of bound states in infinite waveguides has been investigated thoroughly in an elegant paper by Goldstone and Jaffe [13].

The situation is extremely similar for the toroidal geometry considered here. Construct $V(\theta)$ using the same

ansatz for the wave function, and set the zero of the energy as the maximum value of $V(\theta)$. Assuming a constant longitudinal wave function, an elementary variational calculation immediately shows that the ground state energy of this system is necessarily negative if $V(\theta)$ is non-uniform. There will always be classically allowed and forbidden regions with the usual attendant exponential enhancement or suppression of the ground state wave function.

For zero or weak interactions, the order parameter for the atomic system is simply the eigenfunction of the lowest energy state, and it is exponentially localized/delocalized, too. Thus, even small irregularities in $V(\theta)$ in such one-dimensional systems render instability to an inhomogeneous state inevitable, provided only that the interaction is not too strong. Contrary to the quantum phase transition induced by an attractive effective interaction in a uniform toroidal trap, irregularities require a sufficiently strong and repulsive interaction for the density to be (nearly) homogeneous. Furthermore, while the density variation induced by an attractive interaction in the uniform case is sinusoidal close to the transition point, irregularities in $V(\theta)$ result in an exponential localization/delocalization, which is generally more pronounced.

III. MODEL

To demonstrate these effects quantitatively, we consider the following form of $V(\theta)$:

$$V(\theta) = \begin{cases} V_0 & \text{for } |\theta| > \pi/10 \\ 0 & \text{for } |\theta| < \pi/10, \end{cases} \tag{1}$$

which corresponds to a broadening of the torus over 10% of its circumference. We assume the usual contact potential for the atom-atom interaction, $V_{\rm int}(\mathbf{r}-\mathbf{r}') = U_0\delta(\mathbf{r}-\mathbf{r}')$ with $U_0 = 4\pi\hbar^2 a/M$. Here, a is the scattering length for elastic atom-atom collisions and M is the atomic mass. The Hamiltonian of the quasi-one-dimensional system thus becomes

$$\begin{aligned} H/N = &- \frac{\hbar^2}{2MR^2}\int \Psi^\dagger(\theta)\frac{\partial^2}{\partial\theta^2}\Psi(\theta)\,d\theta + \\ &+ \int \Psi^\dagger(\theta)V(\theta)\Psi(\theta)\,d\theta + \\ &+ \frac{2\pi\hbar^2 Na}{MRS}\int \Psi^\dagger(\theta)\Psi^\dagger(\theta)\Psi(\theta)\Psi(\theta)\,d\theta. \end{aligned} \tag{2}$$

Here $N \gg 1$ is the number of atoms. The prefactor of the last term, $2\pi\hbar^2 Na/(MRS)$, is equal to $\pi n U_0$, where n is the (three-dimensional) density, $n = N/(2\pi RS)$, R is the radius of the torus, and S its cross section, with $\sqrt{S} \ll R$. The three energy scales that appear in our problem, are then (i) the energy for motion along the torus $\hbar^2/(2MR^2)$, (ii) the depth of the potential V_0, and (iii) the interaction energy between the atoms, nU_0.

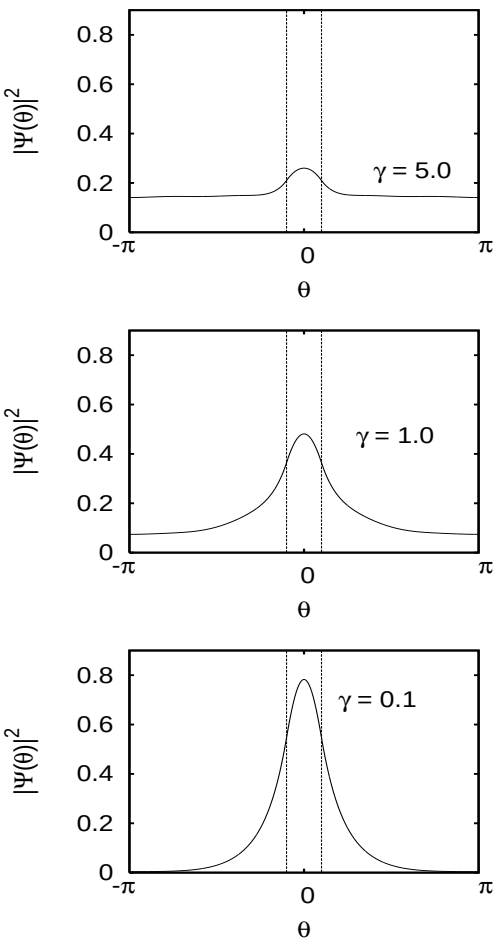

FIG. 1: The density $|\Psi(\theta)|^2$ of Eq. (4) for three values of the coupling constant $\gamma = 0.1, 1.0$, and 5.0, and for $V(\theta)$ given by Eq. (1). The two vertical lines show the range of the potential $V(\theta)$.

For $V_0 = 0$, one finds an instability towards the formation of a localized state induced by an attractive interaction when $n|U_0| \sim \hbar^2/(2MR^2)$. More precisely, the critical value of the ratio, γ, between the potential energy nU_0 and the kinetic energy $\hbar^2/(2MR^2)$, is $\gamma = 4NaR/S = -1/2$ [11, 12]. Finally, the crossover between the phases of localized and homogeneous density that we consider here is given roughly by the condition $nU_0 \sim V_0$.

IV. RESULTS

A. Lowest state

To determine the order parameter $\Psi(\theta)$, we first solve the eigenvalue problem $H_0\psi_m = E_m\psi_m$, where H_0 is the Hamiltonian of the non-interacting problem. The eigenstates ψ_m are parity eigenstates. The actual value of V_0 that we choose is $50/\pi^2$ in units of $\hbar^2/(2MR^2)$,

and it has only one bound state with an energy $E_0 \approx 3.503\hbar^2/(2MR^2)$. Except for this lowest ($m = 0$) eigenvalue, the eigenvalues of all the excited states are quite close to those of the undistorted torus, i.e.,

$$E_m \approx V_0 + \frac{\hbar^2}{2MR^2}\, m^2. \tag{3}$$

Clearly, this expression becomes more accurate for larger values of m.

Having solved the eigenvalue problem, we then expand the order parameter in the eigenstates ψ_m,

$$\Psi(\theta) = \sum_m c_m \psi_m(\theta), \tag{4}$$

and calculate the coefficients c_m variationally. For each value of the coupling constant γ considered, we minimize the expectation value of the Hamiltonian subject to the constraint $\sum_m |c_m|^2 = 1$ corresponding to a fixed number of atoms.

Figure 1 shows $|\Psi(\theta)|^2$ for three values of $\gamma = 0.1, 1.0$, and 5.0, where we kept the lowest five states (of even parity – only even parity states contribute to Ψ). The highest eigenstate, with quantum number m_0, that must be included in the expansion of the order parameter has to satisfy the condition $E_{m_0} \gg nU_0$. This condition is comfortably met even for the largest interaction strength $\gamma = 5.0$ considered, since $E_5 \approx 29.620\hbar^2/(2MR^2)$.

For small values of the interaction energy, $nU_0 \ll E_0$, the dominant component of the order parameter is the lowest eigenstate of the single-particle problem. This is an exponentially decaying state in the classically forbidden regions of the torus, $|\theta| > \pi/10$. Clearly, a similar result will be obtained for any non-uniform choice of $V(\theta)$. As the coupling increases, the density distribution becomes wider. For sufficiently strong interactions, $nU_0 \gg V_0$, the density $|\Psi(\theta)|^2$ becomes homogeneous, reaching the limiting value $1/(2\pi)$.

B. Excitation spectrum

The excitation spectrum is also affected as one goes from the limit of zero interactions (with the order parameter being a localized state) to very strong interactions (with the order parameter being a homogeneous state). For weak interactions, the excitations are single-particle like. Atoms are excited from the eigenstate of the non-interacting problem with quantum number $m = 0$ to some excited state with quantum number $m \neq 0$. In the absence of interactions the corresponding excitation energies are approximately equal to the quadratic law of Eq. (3).

For weak interactions ($\gamma \ll 1$) the excitation spectrum can be calculated using perturbation theory. The total energy of the lowest state is

$$\mathcal{E}_0 \;=\; NE_0 + \frac{2\pi\hbar^2 a}{MRS} N(N-1)\, I_{0000}, \tag{5}$$

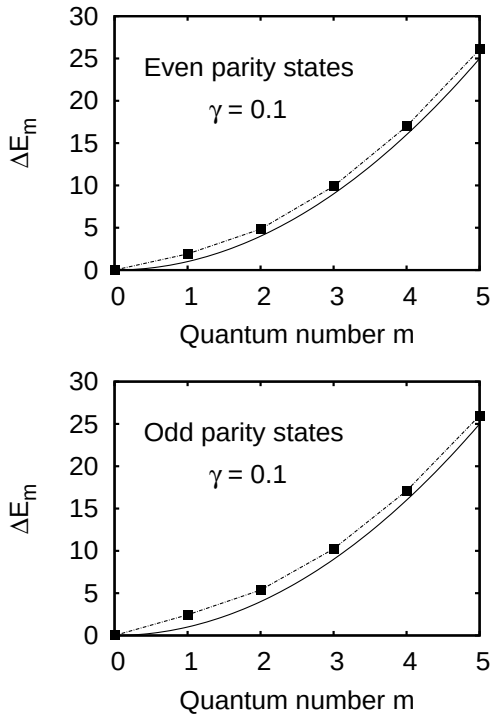

FIG. 2: The points (that are connected with straight, dashed lines) show the excitation spectrum, Eq. (7), for $\gamma = 0.1$ for the lowest five excited states of even and odd parity. The solid lines shows the free-particle excitation energy $E_m - E_0$. The energy is measured in units of $\hbar^2/(2MR^2)$.

where $I_{ijkl} = \int \psi_i^* \psi_j^* \psi_k \psi_l \, d\theta$. Exciting one atom to the state with quantum number m, the total energy becomes

$$\begin{aligned} \mathcal{E}_m \;&=\; (N-1)E_0 + E_m + \\ &+\; \frac{2\pi\hbar^2 a}{MRS}[(N-1)(N-2)\, I_{0000} + 4(N-1)\, I_{0m0m}]. \end{aligned} \tag{6}$$

The factor of four in the last term comes from the direct and the exchange terms of the interaction energy for identical bosons. From Eqs. (5) and (6) we get for $\Delta\mathcal{E}_m = \mathcal{E}_m - \mathcal{E}_0$ that

$$\Delta\mathcal{E}_m = E_m - E_0 + 2\pi\gamma \frac{\hbar^2}{2MR^2}[2\, I_{0m0m} - I_{0000}]. \tag{7}$$

Figure 2 shows $\Delta\mathcal{E}_m$ for the five lowest excited eigenstates of the non-interacting system of even and odd parity, for $\gamma = 0.1$. The excitation energies of the first two excited states (i.e., the even and odd parity states with $|m| = 1$) are $\approx 1.907\hbar^2/(2MR^2)$ and $\approx 2.443\hbar^2/(2MR^2)$, respectively. Clearly for large enough $|m|$ the excitation spectrum is quadratic. On the other hand, for the low-lying excited states there are deviations from the quadratic dependence due to the interactions, and also due to the deviations of the eigenenergies from the formula of Eq. (3), which are more pronounced for low m.

The opposite limit of strong interactions, $\gamma \gg 1$, is more straightforward. Even in the presence of some $V(\theta)$ the density tends towards homogeneity in this limit (as shown in the top panel of Fig. 1). The excitation spectrum of the homogeneous state, $|\Psi(\theta)|^2 = 1/(2\pi)$, is [11, 12]

$$\Delta\mathcal{E}_m = \frac{\hbar^2}{2MR^2}\sqrt{m^2(m^2+2\gamma)}. \quad (8)$$

For $\gamma \gg 1$ the above formula implies that $\Delta\mathcal{E}_m$ scales as $|m|$ in our problem,

$$\Delta\mathcal{E}_m = \frac{\hbar^2}{2MR^2}\sqrt{2\gamma}\,|m|. \quad (9)$$

The above equation implies a speed of sound $c = (R/\hbar)\,\partial\Delta\mathcal{E}_m/\partial m = \hbar\sqrt{2\gamma}/(2MR)$, or $Mc^2 = nU_0$. This result can also be obtained from the energy per unit length, $\epsilon(\sigma)$, according to the formula $Mc^2 = \sigma\,\partial^2\epsilon/\partial\sigma^2$, where $\sigma = N/(2\pi R)$ is the atom density per unit length. In the limit of strong interactions, where $|\Psi|^2 = 1/(2\pi)$, the quadratic contribution to $\epsilon(\sigma)$ is given by the interaction energy, i.e., the last term in the Hamiltonian H of Eq. (2), $\epsilon(\sigma) = 2\pi\hbar^2 a\sigma^2/(MS)$. Thus, $c = \hbar\sqrt{2\gamma}/(2MR)$, in agreement with Eq. (9).

V. DISCUSSION AND CONCLUSIONS

Depending on the various parameters and on the experimental demands in constructing axially-symmetric toroidal traps, it is possible to explore relevant parts of the phase diagram of quasi one-dimensional atomic gases. We have pointed out, however, that the ground state of an atom in a toroidal trap will always have some degree of exponential localization whenever the trapping potential deviates from perfect uniformity. Although we have illustrated this fact with one specific form of $V(\theta)$, the result is general. Even apparently small modulations in $V(\theta)$ can give rise to the localization demonstrated here and have the potential to obscure the quantum phase transition described above for an effective attractive interaction between the atoms. Furthermore, the excitation spectrum is also affected by the presence of some variation in $V(\theta)$. More specifically, this is free-particle-like for zero or sufficiently weak interactions, and it is phonon-like for sufficiently strong interactions.

Our study leads to three conclusions: (i) In realistic toroidal traps, even weak irregularities can result in a density distribution that is exponentially localized. Care must therefore be exercised in comparing experimental results with theoretical predictions assuming a uniform trapping potential. (ii) The excitation spectrum (and thus the dynamics) of the gas is also affected by variations in $V(\theta)$, and caution is again called for. (iii) The controlled modification of the toroidal potential and/or tuning of the coupling constant of the atomic interaction permit engineering of the shape of the atom density in a rather dramatic way. For example, the creation in the torus of a potential resembling our $V(\theta)$ (e.g., using a laser) can change the density distribution from almost homogeneous to highly localized. Since such modifications of toroidal traps can be made with relative ease and that the coupling between the atoms is easily tunable via Feshbach resonances, our results may have useful applications.

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